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G-pre-invex functions in mathematical programming

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Abstract

In the present paper, we introduce the concept of G-pre-invex functions with respect to η defined on an invex set with respect to η. These function unify the concepts of nondifferentiable convexity, pre-invexity and r-pre-invexity. Furthermore, relationships of G-pre-invex functions to various introduced earlier pre-invexity concepts are also discussed. Some (geometric) properties of this class of functions are also derived. Finally, optimality results are established for optimization problems under appropriate G-pre-invexity conditions.

MSC: 26B25; 90C26

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1. Introduction

The notion of convexity undoubtedly plays a dominant role in almost all aspect of mathematical programming. In fact, there are a number of nonlinear programming problems whose objective and constraints functions are nonconvex. Therefore, in the recent years attempts are made by several authors to define various nonconvex classes of functions and to study their optimality criteria in solve such types of problems. One of such a generalization of a convex function is invexity notion introduced in [9] and called in [8]. Over the years, many generalizations of this concept have been given in the literature. For example, in a more general case, Ben-Israel and Mond [7] considered a class of nondifferentiable functions that were called pre-invex in [13] as a generalization of convexity. Weir and Mond [14] studied how and where pre-invex functions can replace convex functions in multiobjective programming problems. The concept of pre-invexity of functions was also generalized to B-pre-invex functions in [12]. Using the definition of a weighted r-mean (where r is a real number) for a sequence of positive numbers, Antczak [1] introduced new classes of (nonconvex) functions and called them (p, r)-pre-invex with respect to η. Moreover, he also considered its subclass of nonconvex functions and called it a class of r-pre-invex functions with respect to η [4]. The class of (p, r)-pre-invex functions with respect to η is an extension of the class of pre-invex functions with respect to η introduced in [7]. Moreover, Antczak [1] gave a characterization of geometric properties of (p, r)-pre-invexity. In [2], Antczak gave a comparative characterization of several distinct pre-invex concepts: pre-invex, B-vex, B-pre-invex, pre-univex, (p, r)-pre-invex. This characterization contains: direct relationships between considered classes of functions, a geometric characterization in terms of distinct

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invex sets, level sets, epigraphs. Further, he proved the fundamental property of pre-invexity concept, i.e., every local minimum of a function of this type is also its global minimum, and he also showed that a set of global minimum points is a p-invex set (in mostly cases, with $p = 0$).

In this paper, we introduce a new class of nonconvex, not necessarily differentiable, functions, called G-pre-invex functions. These functions are pre-invex transformable by a continuous increasing function. Thus, we extend a pre-invexity notion since the defined class of functions contains many various pre-invexity concepts. We also give some "tools", by the help which we are in position to judge that a given function belongs to a class of G-pre-invex functions (with respect to some functions η and G). This result shows that a determination of the satisfaction of G-pre-invexity for a function can be achieved via an intermediate-point G-pre-invexity check. For some classes of nonconvex functions, moreover, we give examples of functions η and G satisfying the inequality from the definition of G-pre-invexity notion. As follows at least from these examples, in some cases of nonconvex functions, the introduced G-pre-invexity notion is more useful than pre-invexity.

Furthermore, we also introduce a definition of a G-invex set with respect to η, which is an extension of a definition of $(0, r)$-invex set introduced earlier in [1]. Making use of the definition of a level set, an epigraph of a function and an introduced definition of a G-invex set with respect to η, we shall give a characterization of geometric properties of G-pre-invexity. Further, a characterization of the fundamental properties (not only geometric) of the introduced class of generalized convex functions is dealt with. We show that the class of functions which are characterized by G-pre-invexity possesses a principal property which has been the base of pre-invexity theory, i.e., any local minimum of these functions is a global minimum. Some of their properties are obtained on the lines of r-pre-invex functions [4], however, established properties extend those well-known for various classes of pre-invex functions.

The further part of considerations is devoted to the optimality results in nondifferentiable constrained mathematical programming problems. We investigate some properties of the solutions of mathematical programming problems with inequality constraints involving functions belonging to the class of functions introduced in this paper. Thus, a number of optimality results are established for optimization problems by assuming the functions involved to be G-pre-invex with respect to the same function η, but with respect to, not necessarily, the same function G.

2. *G*-pre-invex functions

We shall also use a definition of an invex set with respect to η. The definition of a set of this type was given in [7] and subsequently studied by many authors including Antczak [1,2,4], Mohan and Neogy [10], Pini [11].

Definition 1. Let X be a nonempty subset of R^n, $\eta : X \times X \to R^n$ and u be an arbitrary point of X. Then, the set X is said to be invex at u with respect to η if, for each $x \in X$ and any $\lambda \in [0, 1]$,

$$u + \lambda\eta(x, u) \in X. \tag{1}$$

If the relation (1) is satisfied at any $u \in X$ then X is said to be an invex set with respect to η.

A nontrivial example of an invex set in R^2 can be found in [3].

Definition 2 (*Ben-Israel and Mond [7]*). Let X be a nonempty invex with respect to η subset of R^n. Given function $f : X \to R$ is said to be pre-invex (with respect to η) on X if, the following inequality

$$f(u + \lambda\eta(x, u)) \leqslant \lambda f(x) + (1 - \lambda) f(u) \tag{2}$$

holds for all $x, u \in X$ and any $\lambda \in [0, 1]$.

To prove some results in the paper, we need the so-called Condition C introduced in [10]. Now, for a reader's convenience, we recall this condition.

Condition C: We say that the function $\eta : R^n \times R^n \to R^n$ satisfies Condition C if, for any $x, u \in R^n$, the following relations

$$\eta(u, u + \lambda\eta(x, u)) = -\lambda\eta(x, u),$$
$$\eta(x, u + \lambda\eta(x, u)) = (1 - \lambda)\eta(x, u)$$

are satisfied for any $\lambda \in [0, 1]$.

In this section, we shall study functions that are pre-invex transformable by a continuous increasing function. Let *f* be a real-valued continuous function defined on the nonempty invex set *X* (with respect to η), and by $I_f(X)$ the range of *f*, that is, the image of *X* under *f*.

Definition 3. Let *X* be a nonempty invex (with respect to η) subset of R^n. A function $f : X \to R$ is said to be (strictly) *G*-pre-invex at *u* on *X* with respect to η if there exist a continuous real-valued increasing function $G : I_f(X) \to R$ and a vector-valued function $\eta : X \times X \to R^n$ such that, for all $x \in X$ $(x \neq u)$ and any $\lambda \in [0, 1]$,

$$f(u + \lambda\eta(x, u)) \leqslant G^{-1}(\lambda G(f(x)) + (1 - \lambda)G(f(u))) \quad (<). \tag{3}$$

If (3) is satisfied for any $u \in X$ then *f* is (strictly) *G*-pre-invex on *X* with respect to η.

Remark 4. In order to define an analogous class of (strictly) *G*-pre-incave functions with respect to η, the direction of the inequality in the definition of these functions should be changed to the opposite one.

It should be noted that the symmetry between convexity and concavity, and also between pre-invexity and pre-incavity, does not hold for *G*-pre-invex functions with respect to η and *G*-pre-incave functions with respect to η. That is, if *f* is a *G*-pre-invex function with respect to η then $-f$ is not necessarily a *G*-pre-incave function with respect to the same function η; it is, however, $\widetilde{G}$-pre-incave with respect to η with $\widetilde{G}(t) = -G(-t)$.

Now we give an useful lemma, which proof is omitted in the paper.

Lemma 5. *G^{-1} is increasing if and only if G is increasing.*

Now, we present relationships between the introduced class of *G*-pre-invex and other classes of generalized convex functions.

Remark 6. In the case when $\eta(x, u) = x - u$, we obtain a definition of a nondifferentiable *G*-convex function (see [6]).

Remark 7. Every pre-invex function with respect to η introduced in [7] is *G*-pre-invex with respect to the same function η, where $G : I_f(X) \to R$ is defined by $G(a) \equiv a$.

Remark 8. Every *r*-pre-invex function with respect to η with $r > 0$ introduced in [1,2] is *G*-pre-invex with respect to the same function η, where $G : I_f(X) \to R$ is defined by $G(a) = e^{ra}$.

Remark 9. Let *f* be a differentiable function on *X* and *G* be a differentiable function on $I_f(X)$. In this case, Antczak [5] established that any *G*-pre-invex function on *X* with respect to η is also *G*-invex on *X* with respect to the same function η. The converse result is, in general, not true. This means that there exist *G*-invex functions with respect to η which are not *G*-pre-invex function with respect to the same function η. However, Antczak proved that if the function η is assumed to satisfy the so-called Condition C given in [10], then converse result is true. Thus, if *f* is *G* invex with respect to η on a nonempty invex (with respect to η) set *X* and η satisfies Condition C then *f* is *G*-pre-invex function on *X* with respect to the same function η.

Now, we give a necessary and sufficient condition for a continuous function f to be *G*-pre-invex function with respect to η on a nonempty invex (with respect to η) set $X \subset R^n$.

Theorem 10. *Let X be a nonempty invex with respect to η compact subset of R^n. Further, assume that $f : X \to R$ is a continuous function satisfying the following inequality*:

$$f(u + \eta(x, u)) \leqslant f(x) \tag{4}$$

for all $x, u \in X$, the function η satisfies Condition C and there exists an increasing continuous function $G : I_f(X) \to R$. Then f is a G-pre-invex function with respect to η on X if and only if, for any $x, u \in X$, there exists $\theta \in (0, 1)$ such that *the following inequality*

$$f(u+\theta\eta(x,u)) \leqslant G^{-1}(\theta G(f(x)) + (1-\theta)G(f(u))) \tag{5}$$

holds.

Proof. The necessity follows directly from the definition of a G-pre-invex function (see Definition 3). Therefore, we only prove the sufficiency of this theorem.

By means of contradiction, we suppose that there exist $x, u \in X$ and $\widehat{\lambda} \in (0, 1)$ such that

$$f(u+\widehat{\lambda}\eta(x,u)) > G^{-1}(\widehat{\lambda}G(f(x)) + (1-\widehat{\lambda})G(f(u))). \tag{6}$$

We define

$$x_\alpha = u + \alpha\eta(x,u), \quad \alpha \in [0, \widehat{\lambda})$$

and

$$\Omega = \{x_\alpha \in X : f(x_\alpha) = f(u+\alpha\eta(x,u)) \leqslant G^{-1}(\alpha G(f(x)) + (1-\alpha)G(f(u))) \text{ for } \alpha \in [0, \widehat{\lambda})\}.$$

Note that the set Ω is not empty. Indeed, by the definition of Ω, we have, for $\alpha = 0$, $f(x_0) = f(u) \leqslant f(u)$. This means that $x_0 \in \Omega$. We now denote by

$$\gamma = \sup\{\alpha \in [0, \widehat{\lambda}) : x_\alpha \in \Omega\}.$$

Therefore, by (6), it follows that

$$x_\alpha \notin \Omega \quad \text{for } \gamma < \alpha \leqslant \widehat{\lambda}.$$

Thus, there exists a sequence $\{\alpha_n\}$ with $\alpha_n \leqslant \gamma$ and $x_{\alpha_n} \in \Omega$ such that it converges to γ whenever n converges to ∞, that is, $\alpha_n \to \gamma$ when $n \to \infty$. By assumption, f is a continuous function on X and G is a continuous function on a compact set $I_f(X)$. Therefore, G^{-1} is a continuous function. Hence, also by $x_{\alpha_n} \in \Omega$,

$$\begin{aligned} f(x_\gamma) &= \lim_{n\to\infty} f(x_{\alpha_n}) \leqslant \lim_{n\to\infty} G^{-1}(\alpha_n G(f(x)) + (1-\alpha_n)G(f(u))) \\ &= G^{-1}(\gamma G(f(x)) + (1-\gamma)G(f(u))). \end{aligned} \tag{7}$$

Therefore, by the definition of the set Ω follows that

$$x_\gamma \in \Omega.$$

In a similar way, we define

$$u_\alpha = u + \alpha\eta(x,u), \quad \alpha \in (\widehat{\lambda}; 1]$$

and

$$\Gamma = \{u_\alpha \in X : f(u_\alpha) = f(u+\alpha\eta(x,u)) \leqslant G^{-1}(\alpha G(f(x)) + (1-\alpha)G(f(u))) \text{ for } \alpha \in (\widehat{\lambda}; 1]\}.$$

Note that the set Γ is not empty. Indeed, for $\alpha = 1$, by the definition of the set Γ, we have $f(u_1) = f(u+\eta(x,u)) \leqslant f(x)$. Therefore, by assumption (4) follows that $u_1 \in \Gamma$. We now denote by

$$\beta = \inf\{\alpha \in (\widehat{\lambda}; 1] : u_\alpha \in \Gamma\}.$$

Then, by (6), it follows that

$$u_\alpha \notin \Gamma \quad \text{for } \widehat{\lambda} \leqslant \alpha < \beta.$$

Thus, there exists a sequence $\{\alpha_n\}$ with $\alpha_n \geqslant \beta$ and $u_{\alpha_n} \in \Gamma$ such that it converges to β whenever n converges to ∞, that is, $\alpha_n \to \beta$ when $n \to \infty$. By assumption, f is a continuous function on X. Since G^{-1} is a continuous function then using $u_{\alpha_n} \in \Gamma$, we get

$$\begin{aligned} f(u_\beta) &= \lim_{n\to\infty} f(x_{\alpha_n}) \leqslant \lim_{n\to\infty} G^{-1}(\alpha_n G(f(x)) + (1-\alpha_n)G(f(u))) \\ &= G^{-1}(\beta G(f(x)) + (1-\beta)G(f(u))). \end{aligned} \tag{8}$$

Therefore, by the definition of the set Γ follows that

$$u_\beta \in \Gamma.$$

Thus, by the definitions of β and γ, we have

$$0 \leqslant \beta < \widehat{\lambda} < \gamma \leqslant 1.$$

Now, for any $\lambda \in (0, 1)$, we establish that $u_\beta + \lambda\eta(x_\gamma, u_\beta) = u + (\lambda\gamma + (1-\lambda)\beta)\eta(x, u)$. Indeed, by using Condition C, we have

$$\begin{aligned} u_\beta + \lambda\eta(x_\gamma, u_\beta) &= u + \beta\eta(x, u) + \lambda\eta(u + \gamma\eta(x, u), u + \beta\eta(x, u)) \\ &= u + \beta\eta(x, u) + \lambda\eta(u + \gamma\eta(x, u), u + \gamma\eta(x, u) + (\beta-\gamma)\eta(x, u)) \\ &= u + \beta\eta(x, u) + \lambda\eta\left(u + \gamma\eta(x, u), u + \gamma\eta(x, u) + \frac{\beta-\gamma}{1-\gamma}\eta(x, u + \gamma\eta(x, u))\right) \\ &= u + \beta\eta(x, u) - \lambda\frac{\beta-\gamma}{1-\gamma}\eta(x, u + \gamma\eta(x, u)) \\ &= u + (\lambda\gamma + (1-\lambda)\beta)\eta(x, u). \end{aligned}$$

Thus, for any $\lambda \in (0, 1)$,

$$f(u_\beta + \lambda\eta(x_\gamma, u_\beta)) = f(u + (\lambda\gamma + (1-\lambda)\beta)\eta(x, u)). \tag{9}$$

By assumption, G is a continuous increasing function on $I_f(X)$. Hence, using (6) together with the definitions of β and γ, it follows that

$$f(u + (\lambda\gamma + (1-\lambda)\beta)\eta(x, u)) > G^{-1}((\lambda\gamma + (1-\lambda)\beta)G(f(x) + (1 - (\lambda\gamma + (1-\lambda)\beta)G(f(u)). \tag{10}$$

Hence, for any $\lambda \in (0, 1)$,

$$\begin{aligned} &G^{-1}((\lambda\gamma + (1-\lambda)\beta)G(f(x)) + (1 - (\lambda\gamma + (1-\lambda)\beta)G(f(u)) \\ &\quad = G^{-1}(\lambda[\gamma G(f(x)) + (1-\gamma)G(f(u))] + (1-\lambda)[\beta G(f(x)) + (1-\beta)G(f(u))]). \end{aligned} \tag{11}$$

By assumption, G is an increasing function. Then, by Lemma 5, G^{-1} is also increasing. Therefore, by (7) and (8), we get, for any $\lambda \in (0, 1)$,

$$\begin{aligned} &G^{-1}(\lambda[\gamma G(f(x)) + (1-\gamma)G(f(u))] + (1-\lambda)[\beta G(f(x)) + (1-\beta)G(f(u))]) \\ &\quad \geqslant G^{-1}(\lambda G(f(x_\gamma)) + (1-\lambda)[G(f(u_\beta))]). \end{aligned} \tag{12}$$

Then, by (9)–(12), we obtain that the following inequality

$$f(u_\beta + \lambda\eta(x_\gamma, u_\beta)) > G^{-1}(\lambda G(f(x_\gamma)) + (1-\lambda)G(f(u_\beta)))$$

holds for any $\lambda \in (0, 1)$. This is a contradiction to (5). Thus, the sufficiency of theorem is proved. $\square$

The following corollary follows directly from the above theorem.

Corollary 11. *Let X be a nonempty invex with respect to η compact subset of R^n. Further, assume that $f : X \to R$ is a continuous function satisfying the following inequality*:

$$f(u+\eta(x,u)) \leqslant f(x)$$

for all $x, u \in X$, the function η satisfies Condition C and there exists an increasing continuous function $G : I_f(X) \to R$. Then, f is a G-pre-invex function with respect to η on X if and only if, there exists $\theta \in (0,1)$ such that, for any $x, u \in X$, the following inequality

$$f(u+\theta\eta(x,u)) \leqslant G^{-1}(\theta G(f(x)) + (1-\theta)G(f(u)))$$

holds. Now, we establish some of the basic properties of *G*-pre-invex functions.

Proposition 12. *Let X be a nonempty invex set with respect to η subset of R^n and let $f_i : X \to R, i \in I$, be a finite or infinite collection of G-pre-invex function with respect to the same function η and G on X. Define $f(x) = \sup\{f_i(x) : i \in I\}$, for every $x \in X$. Further, assume that for every $x \in X$, there exists $i^* = i(x) \in I$, such that $f(x) = f_{i^*}(x)$. Then f is a G-pre-invex function with respect to the same function η on X.*

Proof. Suppose, contrary to the result, that *f* is not *G*-pre-invex with respect to η on *X*. Then there exist $x, u \in X$ and $\lambda \in [0,1]$ such that

$$f(u+\lambda\eta(x,u)) > G^{-1}(\lambda G(f(x)) + (1-\lambda)G(f(u))). \tag{13}$$

We denote $z = u + \lambda\eta(x,u)$. From the assumptions of the theorem, there exist $i(z) := i_z \in I$, $i(x) := i_x \in I$, $i(u) := i_u \in I$, satisfying

$$f(z) = f_{i_z}(z), \quad f(x) = f_{i_x}(x), \quad f(u) = f_{i_u}(u). \tag{14}$$

Therefore, by (13),

$$f_{i_z}(z) > G^{-1}(\lambda G(f_{i_x}(x)) + (1-\lambda)G(f_{i_u}(u))). \tag{15}$$

By assumption, any f_i, $i \in I$, is *G*-pre-invex function with respect to *X*. Then, using the *G*-pre-invexity of f_{i_z}, we obtain, by Definition 3,

$$f_{i_z}(z) \leqslant G^{-1}(\lambda G(f_{i_z}(x)) + (1-\lambda)G(f_{i_z}(u))). \tag{16}$$

From the definition of *G*-pre-invexity, *G* is an increasing function on its domain. Then, by Lemma 5, G^{-1} is increasing. Since $f_{i_z}(x) \leqslant f_{i_x}(x)$ and $f_{i_z}(u) \leqslant f_{i_u}(u)$ then (16) gives

$$f_{i_z}(z) \leqslant G^{-1}(\lambda G(f_{i_x}(x)) + (1-\lambda)G(f_{i_u}(u))).$$

The inequality above contradicts (15). □

Proposition 13. *Let f be a G_1-pre-invex function with respect to η on a nonempty invex set $X \subset R^n$ with respect to η and G_2 be a continuous increasing function on $I_f(X)$. If the function $g(t) = G_2G_1^{-1}(t)$ is convex on the image under G_1 of the range of f, then f is also G_2-pre-invex function with respect to the same function η on X.*

Proof. Let *X* be a nonempty invex subset of R^n with respect to η and we assume that *f* is G_1-pre-invex with respect to η on *X*. Then, by Definition 3, the following inequality

$$f(u+\lambda\eta(x,u)) \leqslant G_1^{-1}(\lambda G_1(f(x)) + (1-\lambda)G_1(f(u))) \tag{17}$$

is satisfied for all $x, u \in X$ and any $\lambda \in [0,1]$. Let G_2 be a continuous increasing function on $I_f(X)$. Hence, by (17),

$$G_2(f(u+\lambda\eta(x,u))) \leqslant G_2G_1^{-1}(\lambda G_1(f(x)) + (1-\lambda)G_1(f(u))).$$

By assumption, $g(t) = G_2 G_1^{-1}(t)$ is convex on the image under G_1 of the range of *f*. Then, by convexity hypothesis of $G_2 G_1^{-1}$, it follows that the following inequality

$$\begin{aligned} &G_2 G_1^{-1}(\lambda G_1(f(x)) + (1-\lambda) G_1(f(u))) \\ &\quad \leqslant \lambda G_2 G_1^{-1}(G_1(f(x))) + (1-\lambda) G_2 G_1^{-1}(G_1(f(u))) \\ &\quad = \lambda G_2(f(x)) + (1-\lambda) G_2(f(u)) \end{aligned}$$

holds for all $x, u \in X$ and any $\lambda \in [0, 1]$. Letting G_2^{-1} denote the inverse of G_2, we get from the above inequality

$$G_1^{-1}(\lambda G_1(f(x)) + (1-\lambda) G_1(f(u))) \leqslant G_2^{-1}(\lambda G_2(f(x)) + (1-\lambda) G_2(f(u))). \tag{18}$$

Thus, using (17) together with (18), we obtain that the inequality

$$f(u + \lambda \eta(x, u)) \leqslant G_2^{-1}(\lambda G_2(f(x)) + (1-\lambda) G_2(f(u)))$$

is satisfied for all $x, u \in X$ and any $\lambda \in [0, 1]$. This by Definition 3 means that *f* is a G_2-pre-invex function with respect to η on *X*. □

Corollary 14. *Let f be G-pre-invex on a nonempty invex set $X \subset R^n$ with respect to η. If G is concave on $I_f(X)$ then f is pre-invex with respect to the same function η on X.*

Proof. Let *y* and *z* be two points $I_f(X)$. From the assumption *G* is concave on $I_f(X)$. Thus, the following inequality

$$G(\lambda y + (1-\lambda) z) \geqslant \lambda G(y) + (1-\lambda) G(z)$$

holds for any $\lambda \in [0, 1]$. Let $G(y) = x$ and $G(z) = u$. Then, for each pair of points *x* and *u* in the image *G* of $I_f(X)$, that is, $y = G^{-1}(x)$ and $z = G^{-1}(u)$, we have

$$G(\lambda G^{-1}(x) + (1-\lambda) G^{-1}(u)) \geqslant \lambda G(G^{-1}(x)) + (1-\lambda) G(G^{-1}(u)) = \lambda x + (1-\lambda) u. \tag{19}$$

Using both sides of (19) as arguments for G^{-1}, we get

$$G^{-1} G(\lambda G^{-1}(x) + (1-\lambda) G^{-1}(u)) \geqslant G^{-1}(\lambda x + (1-\lambda) u).$$

Thus,

$$\lambda G^{-1}(x) + (1-\lambda) G^{-1}(u) \geqslant G^{-1}(\lambda x + (1-\lambda) u).$$

This means that G^{-1} is convex. Letting $G_1 = G$, $G_2 = t$ then $g(t) = G_2 G_1^{-1}(t)$ is convex. Hence, by Proposition 13, *f* is G_2-pre-invex with respect to η. But G_2 is the identity function; hence *f* is pre-invex with respect to the same function η on *X*. □

Now, we present some classes of functions which are *G*-pre-invex functions.

Proposition 15. *Let X be a nonempty invex set with respect to η subset of R^n and $h : X \to R$ be defined by*

$$h(x) = \Psi(f(x)),$$

where $f : X \to R$ is a convex function on X satisfying $f(0) = 0$ and $\Psi : I_f(X) \to R$ is a continuous increasing function on $I_f(X)$. Then h is G-pre-invex function on X with respect to the function $\eta : X \times X \to R^n$ defined by

$$\eta(x, u) = -u,$$

whereas the function $G : I_f(X) \to R$ is defined as follows:

$$G = \Psi^{-1},$$

where Ψ^{-1} denotes the inverse of Ψ.

To illustrate this result, we give an example of such a G-pre-invex function.

Example 16. Let $X = R$ and a function $h : X \to R$ be defined by

$$h(x) = \arctan(|x^3|).$$

Since the considered function h is the following form $h(x) = \Psi(f(x))$, where $f(x) = |x^3|$ is a convex function on R and $\Psi(t) = \arctan(t)$ is an increasing continuous real-valued function on R then by Proposition 15 follows that h is a G-pre-invex function on R with respect to $\eta : R \times R \to R$ defined by

$$\eta(x, u) = -u,$$

and with respect to the function $G : I_h(X) \to R$ defined by

$$G(t) = \Psi^{-1}(t) = \tan(t).$$

It is not difficult to see that the considered function h is neither G-convex on X [6] nor pre-invex on X with respect to the same function η defined above. Indeed, h is not pre-invex on R with respect to η because

$$f(u + \lambda\eta(x, u)) > \lambda f(x) + (1 - \lambda) f(u), \quad \lambda = 0, 5, \quad u = 1, \quad x = -2.$$

Proposition 17. *Let X be a nonempty invex (with respect to η) subset of R^n. If $f : X \to R$ is a pre-invex function on X with respect to η then f is also a G-pre-invex function with respect to the same function η and with respect to any increasing convex function G.*

The converse result is, in general, not true. This means that there exists a G-pre-invex function with respect to η and with respect to an increasing convex function G, but f is not a pre-invex function with respect to the same function η. Now, we give an example of such a G-pre-invex function.

Example 18. Let $X = R$ and a function $f : X \to R$ be defined by

$$f(x) = \ln(|x| + 1).$$

By Proposition 15, it follows that f is G-pre-invex function on R with respect to the function η defined by

$$\eta(x, u) = -u.$$

Note that f is not pre-invex on X with respect to the same function η because

$$f(u + \lambda\eta(x, u)) > \lambda f(x) + (1 - \lambda) f(u), \quad \lambda = 0, 5, \quad u = 1, \quad x = 0.$$

Further, it is not difficult to show that f is also 1-pre-invex function on R with respect to the function $\widetilde{\eta}$ defined by

$$\widetilde{\eta}(x, u) = \begin{cases} x - u & \text{if } xu < 0, \\ -x - u & \text{if } xu \geqslant 0. \end{cases}$$

Hence, by Remark 8, f is G-pre-invex function on R with respect to the same function $\widetilde{\eta}$, where $G(t) = e^{rt}$ with $r = 1$. Also in this case f is not pre-invex on R with respect to the same function $\widetilde{\eta}$ because

$$f(u + \lambda\eta(x, u)) > \lambda f(x) + (1 - \lambda) f(u), \quad \lambda = 0, 5, \quad u = 1, \quad x = 0.$$

What is more, there exists G-pre-invex function on R with respect to η and with respect to a nonconvex function G. Now, we give an example of such a class of G-pre-invex functions.

Example 19. Let $X = R$ and a function $f : X \to R$ be defined by

$$f(x) = \arctan(k - |x|), \quad \text{where } k \in R. \tag{20}$$

It is not difficult to show, by Definition 3, that any f, defined above, is G-pre-invex on R with respect to the function η^1 defined by

$$\eta^1(x,u)=\begin{cases} x-u & \text{if } xu\geqslant 0,\\ u-x & \text{if } xu<0,\end{cases}$$

where

$$G(t)=\tan(t).$$

Also in this case, there exists more than one function η, with respect to which the considered class of functions is G-pre-invex function on R. Indeed, any functions defined by (20) is also G-pre-invex on R with respect to the function η^2 defined by

$$\eta^2(x,u)=\begin{cases} x-u & \text{if } xu\geqslant 0,\\ -x & \text{if } xu<0.\end{cases}$$

Further, the functions defined by (20) are pre-invex neither with respect to η^1 nor η^2. Note that we cannot use Proposition 15 to show that any functions defined by (20) is G-pre-invex on R since an interior function of the composite function (20) is not convex.

Remark 20. Based on Example 19, it is not difficult to show, by Definition 3, more general result. Indeed, it is not difficult to show that any function $h : X \to R$ defined by $h(x)=\Psi(f(x))$, where X is a nonempty invex (with respect to η) subset of R^n, $f(x)=k-|x|$, $k\in R$, and $\Psi : I_f(X)\to R$ is a continuous increasing function on $I_f(X)$, is G-pre-invex on X with respect to the functions η^1 and η^2 defined above and with respect to the function $G : I_f(X)\to R$ defined by $G=\Psi^{-1}$. What is more, the functions η^1 and η^2 are not, of course, unique functions with respect to functions h defined above are G-pre-invex function on X.

Proposition 21. *Let X be a nonempty invex with respect to η subset X of R^n. Let $h : X\to R$ be defined by*

$$h(x)=\Psi\left(\frac{1}{f(x)}\right),$$

where $f : X\to R$ is a pre-incave function on X with $f(x)>0$ for all $x\in X$ and $\Psi : I_f(X)\to R$ is a continuous increasing function on $I_f(X)$ such that its inverse Ψ^{-1} is convex on R. Then h is a G-pre-invex function on X with respect to the same function η and with respect to the function $G : I_f(X)\to R$ is defined by $G=\Psi^{-1}$, where Ψ^{-1} denotes the inverse of Ψ.

Now, we introduce the definition of a G-invex set with respect to η, which will enable us to give another geometric properties of G-pre-invex functions with respect to η.

Definition 22. Let T be a given subset of $R^n\times R$. Then T is said to be a G-invex set with respect to η if there exist a vector-valued function $\eta : R^n\times R^n\to R^n$ and a continuous real-valued increasing function $G : R\to R$ such that, the relation

$$(x_2+\lambda\eta(x_1,x_2),\,G^{-1}(\lambda G(y_1)+(1-\lambda)G(y_2)))\in T$$

is true for any $(x_1,y_1)\in T$, $(x_2,y_2)\in T$ and any $\lambda\in[0,1]$.

Remark 23. Note that in the case when $G(x)=e^{rx}$, where r is any arbitrary positive real number, then we obtain the definition of a $(0,r)$-invex set with respect to η introduced by Antczak (see [1, Definition 31]).

Theorem 24. *Let X be a nonempty invex with respect to η subset of R^n. If $f : X\to R$ is a G-pre-invex function on X with respect to η then the level set $S_\alpha=\{x\in X : f(x)\leqslant\alpha\}$ is an invex set with respect to η for each $\alpha\in R$.*

Proof. We assume that X is a nonempty invex with respect to η subset of R^n and $f : X \to R$ is a G-pre-invex function on X with respect to η. Let $x, y \in S_\alpha$ for an arbitrary real number α. Then,

$$f(x) \leqslant \alpha \quad \text{and} \quad f(u) \leqslant \alpha.$$

Hence, by Definition 3 and from the definition of G, it follows that

$$f(u + \lambda\eta(x, u)) \leqslant G^{-1}(\lambda G(f(x)) + (1 - \lambda)G(f(u))) \leqslant G^{-1}(G(\alpha)) = \alpha.$$

Then, by the definition of a level set we conclude that $u + \lambda\eta(x, u) \in S_\alpha$ for any $\lambda \in [0, 1]$. Hence, by Definition 1, we conclude that S_α is an invex set with respect to η. □

Theorem 25. *Let X be a nonempty invex with respect to η subset of R^n and f be a real-valued function defined on X. Then f is a G-pre-invex function on X with respect to η if and only if its epigraph $E_f = \{(x, \alpha) : x \in X, \alpha \in R, f(x) \leqslant \alpha\}$ is a G-invex set with respect to η.*

Proof. We assume that X is a nonempty invex with respect to η subset of R^n and $f : X \to R$ is a G-pre-invex function on X with respect to η. Let $(x, \alpha) \in E_f$ and $(u, \beta) \in E_f$. Then $x, u \in X$, $\alpha, \beta \in R$, and

$$f(x) \leqslant \alpha \quad \text{and} \quad f(u) \leqslant \beta. \tag{21}$$

Then using Definition 3 together with (21) we obtain for any $\lambda \in [0, 1]$,

$$\begin{aligned} f(u + \lambda\eta(x, u)) &\leqslant G^{-1}(\lambda G(f(x)) + (1 - \lambda)G(f(u))) \\ &\leqslant G^{-1}(\lambda G(\alpha) + (1 - \lambda)G(\beta)). \end{aligned}$$

By the definition of an epigraph of f, this means that

$$(u + \lambda\eta(x, u), G^{-1}(\lambda G(\alpha) + (1 - \lambda)G(\beta))) \in E_f.$$

Thus, by Definition 22 we conclude that E_f is a G-invex set with respect to η.

Conversely, let E_f be a G-invex set with respect to η. Then, for each $x, u \in X$, we have $(x, f(x)) \in E_f$ and $(u, f(u)) \in E_f$. Then, by hypothesis and the definition of an epigraph of f, the following inequality

$$f(u + \lambda\eta(x, u)) \leqslant G^{-1}(\lambda G(f(x)) + (1 - \lambda)G(f(u)))$$

holds for any $\lambda \in [0, 1]$. This, by Definition 3, implies that f is a G-pre-invex function on X with respect to η. □

Now, we establish a necessary and sufficient condition for $f : X \to R$ to be a G-pre-invex function on X with respect to η.

Theorem 26. *Let X be a nonempty invex with respect to η subset of R^n and f be a real-valued function defined on X. Then f is a G-pre-invex function on X with respect to η if and only if, for each pair of points $x, u \in X$, the following relation*

$$f(u + \lambda\eta(x, u)) \leqslant G^{-1}(\lambda G(f(\alpha)) + (1 - \lambda)G(f(\beta))) \tag{22}$$

is fulfilled for any $\lambda \in [0, 1]$, whenever $f(x) \leqslant \alpha$ and $f(u) \leqslant \beta$.

Proof. We assume that X is a nonempty invex with respect to η subset of R^n and f is a real-valued function defined on X. Let $x, u \in X$ such that

$$f(x) \leqslant \alpha \quad \text{and} \quad f(u) \leqslant \beta. \tag{23}$$

By hypothesis $u + \lambda\eta(x, u) \in X$ for any $\lambda \in [0, 1]$. Since f is a G-pre-invex function on X with respect to η then for any $\lambda \in [0, 1]$,

$$f(u + \lambda\eta(x, u)) \leqslant G^{-1}(\lambda G(f(x)) + (1 - \lambda)G(f(u))).$$

Hence, by (23),

$$f(u+\lambda\eta(x,u))\leqslant G^{-1}(\lambda G(f(\alpha))+(1-\lambda)G(f(\beta))).$$

Conversely, we assume that for each pair of points $x, u \in X$, the relation (22) is fulfilled for any $\lambda \in [0, 1]$, whenever $f(x)\leqslant\alpha$ and $f(u)\leqslant\beta$. We now show that the epigraph of *f* is a *G*-invex set with respect to η. Let $(x,\alpha) \in E_f$ and $(u,\beta) \in E_f$. Then, by the definition of an epigraph, we have that $x, u \in X$ and $f(x)\leqslant\alpha$, $f(u)\leqslant\beta$. Thus, by assumption the relation (22) is fulfilled for any $\lambda \in [0, 1]$. Then, by the definition of an epigraph of *f*, it follows that the relation

$$(u+\lambda\eta(x,u), G^{-1}(\lambda G(f(\alpha))+(1-\lambda)G(f(\beta))))\in E_f$$

is true for any $\lambda \in [0, 1]$. This means by Theorem 25 that *f* is a *G*-pre-invex function on *X* with respect to η. $\square$

Theorem 27. *Let $F \subset R^{n+1}$ defined by $F := \{(x,\alpha) : x \in R^n, \alpha \in R\}$ be any set which is G-invex with respect to η and let $f(x) = \inf\{\alpha : (x,\alpha) \in F\}$. Then f is a G-pre-invex function on X with respect to η.*

Proof. Let $\alpha_1, \beta_1 \in R$ and $x, u \in R^n$ such that

$$f(x)\leqslant\alpha_1 \quad \text{and} \quad f(u)\leqslant\beta_1.$$

Then there exist α_2 and β_2 such that $(x,\alpha_2) \in F$ and $(u,\beta_2) \in F$ and, moreover,

$$f(x)<\alpha_2<\alpha_1 \quad \text{and} \quad f(u)<\beta_2<\beta_1.$$

By assumption, *F* is a *G*-invex set with respect to η. Thus, by Definition 22, the relation

$$(u+\lambda\eta(x,u), G^{-1}(\lambda G(\alpha_2)+(1-\lambda)G(\beta_2)))\in F$$

is satisfied for any $\lambda \in [0, 1]$. By definition of *f* we get, for any $\lambda \in [0, 1]$,

$$f(u+\lambda\eta(x,u))\leqslant G^{-1}(\lambda G(\alpha_2)+(1-\lambda)G(\beta_2)).$$

Hence, by Theorem 26, we get the conclusion of this theorem. $\square$

As it is known [2], a characteristic property of various classes of pre-invex functions with respect to η is the fact that each local minimum of any function belonging to these classes is its global minimum. It turns out that this is also the case for the introduced class of *G*-pre-invex functions (with respect to η). Moreover, the set of global minimum points of a function of this type is an invex set with respect to η.

Theorem 28. *Let $f : X \to R$ be a G-pre-invex function with respect to η on X, and we assume η that satisfies the following condition: $\eta(x,u) \neq 0$ when $x \neq u$. Then each local minimum point of the function f is its point of global minimum. The set of points which are global minima of f is an invex set with respect to η.*

Proof. Assume that $u \in X$ is a local minimum point of *f* which is not a global minimum point. Hence, there exists a point $\widetilde{x} \in X$ such that $f(\widetilde{x}) < f(\bar{x})$. By assumption, *f* is *G*-pre-invex with respect to η on *X*. Thus, by Definition 3, the inequality

$$f(\bar{x}+\lambda\eta(\widetilde{x},\bar{x}))\leqslant G^{-1}(\lambda G(f(\widetilde{x}))+(1-\lambda)G(f(\bar{x}))) \tag{24}$$

holds for any $\lambda \in [0, 1]$. Taking into account the fact that $f(\widetilde{x}) < f(\bar{x})$ for any $0<\lambda<\alpha<1$ for some fixed α, we get

$$f(\bar{x}+\lambda\eta(\widetilde{x},\bar{x}))<G^{-1}(\lambda G(f(\bar{x}))+(1-\lambda)G(f(\bar{x})))=G^{-1}(G(f(\bar{x})))=f(\bar{x}).$$

Thus, we have

$$f(\bar{x}+\lambda\eta(\widetilde{x},\bar{x}))<f(\bar{x})$$

which holds for any $0<\lambda<\alpha<1$. Letting $\lambda \to 0$, we obtain $f(\widetilde{x}) < f(\bar{x})$, a contradiction to the definition of a local minimum at $\bar{x}$.

Denote by A the set of points of global minimum of f and let x and u be arbitrary points belonging to A. In order to prove that A is an invex set with respect to η, we have to show by Definition 1 that, for any $0 \leqslant \lambda \leqslant 1$, the relation $u + \lambda\eta(x, u) \in A$ is true. Since f is G-pre-invex with respect to η then (24) is satisfied. Since $f(x) = f(u)$ (because $x, u \in A$), we have from (24)

$$f(u + \lambda\eta(x, u)) \leqslant G^{-1}(\lambda G(f(u)) + (1 - \lambda)G(f(u))),$$

and, so, for any $\lambda \in [0, 1]$,

$$f(u + \lambda\eta(x, u)) \leqslant G^{-1}(G(f(u))) = f(u) = f(x).$$

Since x and u are points of a global minimum of f, it follows that, for any $\lambda \in [0, 1]$, the following relation

$$u + \lambda\eta(x, u) \in A$$

is satisfied. This by Definition 1 means that A is an invex set with respect to η. $\square$

3. G-pre-invex nonlinear constrained mathematical programming

Now, we consider the following nonlinear mathematical programming problem (P):

$$\begin{aligned} & f(x) \to \min, \\ & g_i(x) \leqslant 0, \quad i \in J = \{1, \ldots, m\}, \qquad \text{(P)} \end{aligned}$$

where $f : X \to R$, $g_i : X \to R$, $i \in J$, and X is a nonempty subset of R^n. We denote the set of all feasible solutions in (P) by

$$D := \{x \in X : g_i(x) \leqslant 0, \ i \in J\}.$$

Theorem 29. *Let the objective function f be G-pre-invex with respect to η on D and, moreover, the constraint functions $g_i, i \in J$, are G_i-pre-invex with respect to the same function η on D. Then, the set of all optimal solutions A is an invex set with respect to η.*

Proof. Let $\overline{x}_1$ and $\overline{x}_2$ be optimal solutions in (P) such that $\overline{x}_1 \neq \overline{x}_2$. Then $f(\overline{x}_1) = f(\overline{x}_2) = \min_{x \in D} f(x)$. By assumption, f is G-pre-invex with respect to η on D. Since $\overline{x}_1, \overline{x}_2 \in D$, therefore, by Definition 3,

$$f(\overline{x}_2 + \lambda\eta(\overline{x}_1, \overline{x}_2)) \leqslant G^{-1}(\lambda G(f(\overline{x}_1)) + (1 - \lambda)G(f(\overline{x}_2))).$$

Thus, by $f(\overline{x}_1) = f(\overline{x}_2)$,

$$f(\overline{x}_2 + \lambda\eta(\overline{x}_1, \overline{x}_2)) \leqslant G^{-1}(\lambda G(f(\overline{x}_1)) + (1 - \lambda)G(f(\overline{x}_2))) = f(\overline{x}_1) = f(\overline{x}_2).$$

To prove that $\overline{x}_2 + \lambda\eta(\overline{x}_1, \overline{x}_2) \in A$ for any $\lambda \in [0, 1]$, it is sufficient to show that $\overline{x}_2 + \lambda\eta(\overline{x}_1, \overline{x}_2) \in D$ for any $\lambda \in [0, 1]$. By assumption, $g_i, i \in J$, are G_i-pre-invex with respect to the same function η on D. Therefore, by Definition 3, for $i \in J$,

$$g_i(\overline{x}_2 + \lambda\eta(\overline{x}_1, \overline{x}_2)) \leqslant G_i^{-1}(\lambda G_i(g_i(\overline{x}_1)) + (1 - \lambda)G_i(g_i(\overline{x}_2))).$$

From the definition of G-pre-invexity, G is an increasing function on its domain. Then, by Lemma 5, G^{-1} is also increasing. Using $\overline{x}_1, \overline{x}_2 \in D$ together with Lemma 5 we obtain, for any $\lambda \in [0, 1]$,

$$G_i^{-1}(\lambda G_i(g_i(\overline{x}_1)) + (1 - \lambda)G_i(g_i(\overline{x}_2))) \leqslant G_i^{-1}(\lambda G_i(0) + (1 - \lambda)G_i g_i(0)) = G_i^{-1}(G_i(g_i(0))) = 0.$$

Thus, for any $\lambda \in [0, 1]$,

$$g_i(\overline{x}_2 + \lambda\eta(\overline{x}_1, \overline{x}_2)) \leqslant 0.$$

This means, by Definition 1, that the set A of all optimal solutions in (P) is invex with respect to η. $\square$

Now, we illustrate the theorem above by a nonconvex optimization problem.

Example 30. We consider the following optimization problem

$$f(x) = \begin{cases} \ln(2-|x|) & \text{if } |x| \leqslant 1 \\ 0 & \text{if } |x| > 1 \end{cases} \to \min,$$

$$g(x) = \arctan(\tfrac{1}{2} - |x|) \leqslant 0.$$

Note that the set of all feasible solutions $D = \{x \in R : -\infty < x \leqslant -\frac{1}{2} \vee \frac{1}{2} \leqslant x < \infty\}$ and the set of all optimal solutions $A = \{x \in R : -\infty < x \leqslant -1 \vee 1 \leqslant x < \infty\}$. It can be proved by Definition 3 that all functions constituting the considered optimization problem are G-pre-invex on R with respect to the same function η, for example, defined by

$$\eta(x,u) = \begin{cases} x-u & \text{if } xu \geqslant 0, \\ -x-u & \text{if } xu < 0 \end{cases} \tag{25}$$

and with respect to not the same function G. Indeed, the objective function f is G_f-pre-invex on R with respect to $G_f(t) = \exp(t) - 1$ and the constraint function g is G_g-pre-invex on R with respect to $G_g(t) = \tan(t)$. Furthermore, it is not difficult to see that the set of all optimal solutions A is not convex. Since all hypotheses of Theorem 29 are fulfilled then the set of all optimal solutions A in the considered optimization problem is invex with respect to the function η defined above. Indeed, it is not difficult to show by Definition 1 that the set of all optimal solutions A in the considered optimization problem is invex with respect to the function η defined by (25).

Theorem 31. *Let $\overline{x} \in D$ be optimal in (P). Moreover, we assume that f is strictly G-pre-invex with respect to η at $\overline{x}$ on D and the constraint functions g_i, $i \in J$, are G_i-pre-invex with respect to the same function η on D. Then, $\overline{x}$ is an unique optimal solution in problem (P).*

Proof. We proceed by contradiction. Let us suppose that there exists $\widetilde{x} \in D$, being an another optimal solution in problem (P). By assumption, the constraint functions g_i, $i \in J$, are G_i-pre-invex with respect to the same function η on D. Using Definition 3 together with $\overline{x} \in D$ and $\widetilde{x} \in D$, we obtain, for all $i \in J$ and any $\lambda \in [0, 1]$,

$$\begin{aligned} g_i(\overline{x} + \lambda\eta(\widetilde{x}, \overline{x})) &\leqslant G_i^{-1}(\lambda G_i(g_i(\widetilde{x})) + (1-\lambda)G_i(g_i(\overline{x}))) \\ &\leqslant G_i^{-1}(\lambda G_i(0) + (1-\lambda)G_i g_i(0)) = 0. \end{aligned}$$

Thus, for all $i \in J$ and any $\lambda \in [0, 1]$,

$$\overline{x} + \lambda\eta(\widetilde{x}, \overline{x}) \in D. \tag{26}$$

By assumption, f is strictly G-pre-invex with respect to η at $\overline{x}$ on D. Therefore, by Definition 3, the following inequality

$$f(\overline{x} + \lambda\eta(x, \overline{x})) < G^{-1}(\lambda G(f(x)) + (1-\lambda)G(f(\overline{x})))$$

holds for any $\lambda \in [0, 1]$ and all $x \in D$, and also for $x = \widetilde{x}$. Then,

$$f(\overline{x} + \lambda\eta(\widetilde{x}, \overline{x})) < G^{-1}(\lambda G(f(\widetilde{x})) + (1-\lambda)G(f(\overline{x}))) = G^{-1}(G(f(\overline{x}))) = f(\overline{x}).$$

By (26), it follows that $\overline{x} + \lambda\eta(\widetilde{x}, \overline{x}) \in D$. Then the inequality above is a contradiction to the optimality of $\overline{x}$ in problem (P). $\square$

Example 32. We consider the following nonconvex optimization problem

$$\exp(\arctan|x|) \to \min,$$

$$\arctan(|x| - 1) \leqslant 0.$$

Note that the set of all feasible solutions $D = \{x \in R : -1 \leqslant x \leqslant 1\}$ and the feasible solution $\overline{x} = 0$ is optimal in the considered optimization problem. It can be proved by Definition 3 that the objective function f is strictly G_f-pre-invex on R with respect to η and $G_f(t)=\tan(\ln(t))$ and the constraint function g is G_g-pre-invex on R with respect to η and $G_g(t)=\tan(t)$, where the function η, for example, is defined by

$$\eta(x,u)=\begin{cases} x-u & \text{if } |x|<|u|, \\ -u & \text{if } |x|\geqslant|x|. \end{cases} \tag{27}$$

Since all hypotheses of Theorem 31 are fulfilled then, as follows from this theorem, the optimal solution $\overline{x}=0$ is unique in the considered optimization problem. Note that in Example 30, the objective function is only G_f-pre-invex on D, but it is not strictly G_f-pre-invex. Therefore, the considered optimization problem has no unique optimal solution, what is more, the set of all optimal solutions is not bounded.

Let $U_\delta(\overline{x})$ denote a neighborhood of $\overline{x}$ of radius δ.

Theorem 33. *Let the set of all feasible solutions D in problem (P) be an invex set with respect to η and $\overline{x}$ be a local minimum in problem (P). Moreover, we assume that for every $\delta>0$ and for every $x\in D$ there exists $\widetilde{\lambda}\in(0,1]$ such that $\overline{x}+\widetilde{\lambda}\eta(x,\overline{x})\in U_\delta(\overline{x})$. If f is strictly G-pre-invex with respect to η at $\overline{x}$ on D then $\overline{x}$ is a strict global minimum in problem (P).*

Proof. By assumption, f is strictly G-pre-invex with respect to η at $\overline{x}$ on D. Then, by Definition 3, the following inequality

$$f(\overline{x}+\lambda\eta(x,\overline{x}))<G^{-1}(\lambda G(f(x))+(1-\lambda)G(f(\overline{x}))) \tag{28}$$

holds for all $x\in D$ and any $\lambda\in[0,1]$. Since $\overline{x}$ is a local minimum in problem (P) then there exists $U_\alpha(\overline{x})$ such that the inequality

$$f(x)\geqslant f(\overline{x}) \tag{29}$$

holds for all $x\in U_\delta(\overline{x})\cap D$.

Now, let x be any arbitrary feasible point for problem (P), such that $x\neq\overline{x}$. Since D is an invex set with respect to η then

$$\overline{x}+\lambda\eta(x,\overline{x})\in D$$

for all $x\in D$ and any $\lambda\in[0,1]$. By assumption, for every $\delta>0$ and for every $x\in D$ there exists $\widetilde{\lambda}\in(0,1]$ such that $\overline{x}+\widetilde{\lambda}\eta(x,\overline{x})\in U_\delta(\overline{x})$. Thus, there exists $\widetilde{\lambda}\in(0,1]$ such that from (29)

$$f(\overline{x}+\widetilde{\lambda}\eta(x,\overline{x}))\geqslant f(\overline{x}). \tag{30}$$

Using (28) together with (30) we obtain

$$\begin{aligned} f(\overline{x})&\leqslant f(\overline{x}+\widetilde{\lambda}\eta(x,\overline{x}))<G^{-1}(\widetilde{\lambda}G(f(x))+(1-\widetilde{\lambda})G(f(\overline{x}))) \\ &\leqslant G^{-1}(\widetilde{\lambda}G(\max\{f(x),\ f(\overline{x})\})+(1-\widetilde{\lambda})G(\max\{f(x),\ f(\overline{x})\})) \\ &=G^{-1}(G(\max\{f(x),\ f(\overline{x})\}))=\max\{f(x),\ f(\overline{x})\}. \end{aligned} \tag{31}$$

Obviously, it is not possible $\max\{f(x),\ f(\overline{x})\}=f(\overline{x})$ since by (31) we get $f(\overline{x})<f(\overline{x})$. Thus, $\max\{f(x),\ f(\overline{x})\}=f(x)$ and (31) implies that the following inequality

$$f(x)>f(\overline{x})$$

holds. Thus, since x is an arbitrary feasible point for problem (P) then we get the conclusion of theorem. □

Theorem 34. *Let the set of all feasible solutions D in problem (P) be an invex set with respect to η and f be a nonconstant G-pre-incave function with respect to η at any optimal solution $\overline{x}$ on D. Then no interior point of D is an optimal solution of (P), or equivalently, any optimal solution $\overline{x}$ in problem (P), if it exists, must be a boundary point of D.*

Proof. If problem (P) has no solution the theorem is trivially true. Let $\overline{x}$ be an optimal solution in problem (P). By assumption, f is nonconstant on D. Then, there exists a feasible point $\widetilde{x}$ such that

$$f(\widetilde{x}) > f(\overline{x}). \tag{32}$$

Let z be an interior point of D. By assumption, D is an invex set with respect to η. Then by Definition 1 there exists $y \in D$ such that for some $\lambda \in [0, 1)$,

$$z = \widetilde{x} + \lambda\eta(y, \widetilde{x}). \tag{33}$$

Since f is a G-pre-incave function with respect to η at $\overline{x}$ on D then

$$\begin{aligned} f(z) = f(\widetilde{x} + \lambda\eta(y, \widetilde{x})) &\geqslant G^{-1}(\lambda G(f(y)) + (1-\lambda)G(f(\widetilde{x}))) \\ &> G^{-1}(\lambda G(f(\overline{x})) + (1-\lambda)G(f(\overline{x}))) = f(\overline{x}). \end{aligned}$$

From the inequality above we conclude that f does not attain its minimum at an interior point z. This completes the proof of theorem. □

4. Conclusion

In the paper, we have introduced a new (not necessarily differentiable) class of generalized convex functions, called G-pre-invex functions. Thus, we have generalized the results previously established for pre-invex functions introduced in [7] and r-pre-invex functions defined in [1,4]. The introduced G-pre-invexity notion is applicable for a larger class of nonconvex functions than r-pre-invexity. Further, as ‘ from the results proved in the paper, G-pre-invexity notion is also more useful tool than pre-invexity to judge in some cases that a given function posses a generalized invexity property. In other words, in some cases, it is easier to find a function η with respect to which a given function is G-pre-invex than pre-invex. Moreover, such a function η has a simpler form in the case of G-pre-invexity than pre-invexity. This property is useful from the practical point of view, in the first place to prove some optimality results for optimization problems involving functions of this type. What is more, for some classes of nonconvex functions, we have given the functions η and G, with respect to which these functions are G-pre-invex on its domain. These results are illustrated in the paper by suitable examples of nonconvex functions.

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